INTEGRAL METHOD OF SOLVING THE STEFAN PROBLEM IN A SEMITRANSPARENT MEDIUM

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Most studies related to phase transitions refer to opaque media. The wide use of semitransparent materials in various branches of science and technology imposes substantial requirements on the technology of obtaining them.

In this connection the calculated radiation of nonstationary radiative-conductive heat transfer (RCHT) is most necessary for the choice of optimal thermal regimes for the growth of high-quality optical crystals. A small number of studies [1-3] is devoted to numerical studies of RCHT in semitransparent media with first kind phase transitions. This fact is related to a number of problems, generated in the solution of the Stefan boundary-value problem, which becomes integrodifferential as related to the integral nature of radiation fluxes.

The mathematical statement of the Stefan problem with explicit phase separation boundaries is written in the form

$$c_i(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_i \frac{\partial T}{\partial x}\right) + F_i(T), \tag{1}$$

$$0 < x < y(t), \quad i = 1, \quad y(t) < x < L, \quad i = 2; \tag{2}$$

$$s_1\lambda_1 \frac{\partial T}{\partial x} = q_1(T), \quad x = 0, \quad s_2\lambda_2 \frac{\partial T}{\partial x} = q_2(T), \quad x = L; \tag{3}$$

$$\lambda_1 \left.\frac{\partial T}{\partial x}\right|_{y(t)-0} - \lambda_2 \left.\frac{\partial T}{\partial x}\right|_{y(t)+0} = \gamma \frac{dy(t)}{dt}; \quad T(y(t), t) = T^*, \ T(0, x) = T_0(x). \tag{4}$$

The conditions (2) include as special cases boundary conditions of the first, second, and third type, since the parameters s_1, s_2 can acquire the values 0, 1 [4]. Here $c_i(T)$, $\lambda_i(T)$ is the heat capacity of the material per unit volume and the thermal conductivity coefficient, $q_i(T)$ are the densities of the resulting fluxes toward the surface boundaries ($i = 1, 2$), $F_i(T)$ is the density of thermal sources, γ is the phase transition heat, and T^* is the melting temperature.

To solve the boundary-value problem (1)-(4) without thermal sources in the presence of a single front, quite effective difference methods were developed with an explicit front extraction [5], which in the case of dependence of the unknown function on several spatial coordinates do not apply even to single-front problems. Moreover, they are not always useful even in the case of the one-dimensional Stefan problem, when heat release sources exist in the medium, leading to "smearing" of the geometrical boundary of the phase transition front. In other words, a whole phase transition region is manifested.

An example was given in [6] of the unimportance of the classical solution of the Stefan problem for the inhomogeneous thermal conductivity problem. In this relation, a number of authors [4, 7] have developed numerical methods of direct computation for solving the Stefan problem without explicit extraction of the phase transition front, based on the "smearing" principle of heat capacity over temperature, which is independent of the number of measurements. The difference schemes of direct computation of the Stefan problem, verified on self-similar solutions, make it possible to obtain highly accurate solutions.

In this study we investigate nonstationary RCHT during melting a semitransparent material, found between two nontransparent boundaries. It is assumed that the solid phase is

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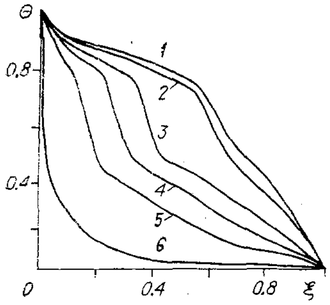

Fig. 1

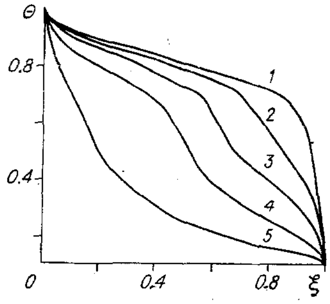

Fig. 2

crystalline, the phase transition temperature is constant, the transition is accompanied by extraction of latent heat of phase transition, there is no convection in the liquid phase, the medium emits and absorbs, but does not scatter thermal energy, the thermophysical properties are independent of temperature, and the optical properties are constant – the medium is gray, the boundary between the phases is diffusely transmitting, while the external surface of the sample are absolutely black.

Taking into account the restrictions mentioned above, the boundary-value problem (1)-(4) with boundary conditions of the first kind in the heat conductivity function $u(\xi, \tau) = \int\limits_0^{\Theta(\xi,\tau)} \Lambda(z)\,dz$ has the dimensionless shape

$$\frac{\partial^2 u}{\partial \xi^2} - u = \frac{\partial H_s(\Theta)}{\partial \tau} + \mathrm{Sk}\frac{\partial \Phi(\Theta)}{\partial \xi} - u, \quad 0 < \xi < 1, \quad \tau > 0; \tag{5}$$

$$u(0, \tau) = \Lambda_1(\Theta(0, \tau) - \Theta^*) + \Lambda_2\Theta^*, \; u(1, \tau) = \Lambda_2\Theta(1, \tau). \tag{6}$$

Conditions (6) were written down with account of the piecewise constant nature of the medium properties (Λ_i, c_i) and the conditions $\Theta(1, \tau) < \Theta^* < \Theta(0, \tau)$. Here $\xi = x/L$: $\Lambda_i = \lambda_i/\lambda_r$; $c_{pi} = c_i/c_r$; $\Theta(x, \tau) = T(x, \tau)/T_r$; $\tau = \lambda_r t/(c_r L^2)$; $\mathrm{Sk} = \sigma_0 T_r^3 L/\lambda_r$; $\Theta^* = T^*/T_r$; $\bar{\gamma} = \gamma/(c_r T_r)$ (i = 1 corresponds to the liquid, and i = 2 corresponds to the solid phase),

$$H_s(\Theta) = \begin{cases} c_{p2}\Theta^* + c_{p1}(\Theta - \Theta^*) + \bar{\gamma} & \text{for} \quad \Theta > \Theta^*, \\ c_{p2}\Theta & \text{for} \quad \Theta < \Theta^*; \end{cases}$$

and r is a subscript referring to the decisive parameters.

The dimensionless radiation flux $\Phi(\Theta)$ is determined by solving the transport equation [8], being an integral relation including the unknown and the boundary temperatures. The expression for $\mathrm{Sk}\partial\Phi/\partial\xi$ in (5) can be treated as a thermal source density, which in the given case depends substantially on the solution of the problem

$$\frac{\partial \Phi(\Theta)}{\partial \xi} = 2\varkappa_i n_i^2 \Big\{ 2\Theta^4(\xi, \tau) - \Theta^4(0, \tau) K_2(\xi\varkappa_i) - \Theta^4(1, \tau) K_2(\varkappa_i(1 - \xi)) - \\ - \varkappa_i \int\limits_0^1 \Theta^4(z, \tau) K_1(\varkappa_i |\xi - z|)\,dz \Big\}, \tag{7}$$

Where $\varkappa_i$, n_i are the absorption and refraction coefficients, and $K_j(\xi) = \int\limits_0^1 \mu^{j-2} \exp(-\xi/\mu)\,d\mu$ are the exponential functions (i = 1, 2; j = 1, 2,...).

Taking into account (7), Eq. (5) becomes nonlinear, being integrodifferential in $\Theta(\xi, \tau)$. After approximating $\partial H_s(\Theta)/\partial\tau$ by a finite-difference scheme by means of the Green's function for the differential operator of the left-hand side (5), the boundary-value problem (5), (6) reduces to the nonlinear integral equation

$$u(\Theta) = (u(0, \tau)\,\mathrm{sh}(1 - \xi) + u(1, \tau)\,\mathrm{sh}\,\xi)/\mathrm{sh}\,1 + \int\limits_0^1 W(\Theta) G(\xi, z)\,dz, \tag{8}$$

$$W(\Theta)=\frac{\Delta H_s(\Theta)}{\Delta\tau}+\mathrm{Sk}\frac{\partial\Phi(\Theta)}{\partial\xi}-u(\Theta),$$

$$G(\xi, z)=\begin{cases}-\operatorname{sh} z\operatorname{sh}(1-\xi)/\operatorname{sh} 1, & z\leqslant\xi,\\ -\operatorname{sh}\xi\operatorname{sh}(1-z)/\operatorname{sh} 1, & z>\xi.\end{cases}$$

In this case the integral is evaluated by Gauss quadratures.

Thus, the Stefan boundary-value problem (1)-(4) with nonlinear internal heat release sources was reduced to a nonlinear integral RCHT equation with phase transitions without explicit extraction of phase separation boundaries. The numerical solution algorithm of Eq. (8) is ideologically equivalent to numerical schemes of direct computation.

The advantage of the suggested method of solving one-dimensional RCHT problems with phase transformations over the finite-difference schemes consists of the fact that, unlike the latter, it is not related to the choice of the accuracy order of the difference scheme in approximating the differential problem by finite differences. The method makes it possible to use effective iteration processes used in solving functional equations. Equation (8) was solved by the iteration method of [9].

The numerical calculation of temperature field formation during the melting of a one-dimensional planar fluorite layer [T = 1700 K, λ_r = 9 W/(m·K)] was carried out for the following dimensionless parameters: $\gamma = -0.1$, $c_1 = 0.75$, $c_2 = 1$, $\Lambda_1 = 2$, $\Lambda_2 = 1$, $n_1 = n_2 = 1.5$, $\varkappa_1 = 2$, $\varkappa_2 = 1$, $\Theta^* = 0.5$, $\Theta(\xi, 0) = \Theta(1, \tau) = 0.1$, $\Theta(0, \tau) = 1$.

The calculation results, shown in Figs. 1 and 2, reflect the dynamic temperature distribution in a semitransparent sample, in which the melting process occurs. Figure 1 shows the dimensionless temperatures profiles in a layer at various moments of dimensionless time $\tau = s\cdot10^{-2}$ (s = 4; 3; 2; 1.5; 1; 0.3, lines 1-6) for Sk = 10 (the radiation-conductive parameter). Figure 2 illustrates the temperature field ($\tau = s\cdot10^{-3}$, lines 1-5 for s = 5, 4, 3, 2, 1) for Sk = 100. As could be expected, the establishment process of the stationary regime starts quite earlier than in the case Sk = 10.

In conclusion we note that the method suggested for solving the Stefan problem has a high degree of convergence of the iteration process.

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